Int. J. Solids Structures Vol. 17, pp. 827–838, 1981
Printed in Great Britain

0020-7683/81/080827-12$02.00/0
Pergamon Press Ltd.

AN APPLICATION OF THE INCLUSION ANALOGY FOR BONDED REINFORCEMENTS

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(*Received* 30 *September* 1980)

Abstract—The problem of estimating the reduction in the crack extension force when a cracked plate is repaired by reinforcing patches bonded to its faces is dealt with in two steps. First, starting with an uncracked plate, the reduction in stress at the prospective location of the crack is determined by treating the reinforced region as an inclusion of higher stiffness than the surrounding plate. Detailed results are given for the case of elliptical orthotropic patches bonded to an infinite orthotropic plate. The second step is to introduce a crack into the reduced stress field prevailing in the plate under the reinforcements. The crack extension force is estimated by using recent results which give an upper bound for the force when the reinforcements cover the whole plate. Some practical implications of the results are discussed.

1. INTRODUCTION

A cracked plate is to sustain a tensile load at right-angles to the crack. Before this load is applied, the plate is repaired by having identical patches bonded to its faces, one on each side, over the crack. The aim of this paper is to estimate the reduction in the crack extension force effected by this repair.

These bonded reinforcements reduce the crack extension force in two distinct ways which are best considered separately. Accordingly, suppose that the plate is not cracked to begin with, but it is reinforced over the prospective location of the crack and loaded. The first effect of the reinforcements is to reduce the stress in the uncracked plate at the prospective location of the crack. The degree of this stress reduction depends mainly on the relative stiffness and on the shape (or aspect ratio) of the reinforcements. Next suppose that a crack is artificially cut into the plate, while the external load is still acting. The second effect of the reinforcements is to restrain the opening of the crack. The effectiveness of this restraint depends primarily on the length of the crack relative to the characteristic length for load transfer from the plate to the reinforcements.

The importance of making the above distinction is that different modelling assumptions can be used in assessing each of those two effects. Thus, for the first effect, it can be assumed that the bond does not allow any relative displacement between the plate and the reinforcements. This rigid-bond assumption ignores the finite width of the load-transfer zone around the boundary of the reinforcements (an idealization which is justified when the actual width of the transfer zone is small relative to the lateral dimensions of the reinforcements). The area covered by the reinforcements can then be treated as an inclusion of higher stiffness than the surrounding plate. This inclusion analogy was formulated by Muki and Sternberg [1] for the case of an isotropic plate and isotropic reinforcements. The principal steps in this formulation are recalled in Section 2. For the present application (which differs from that considered in [1]), it is desirable to extend the formulation to orthotropic constituents, to account for the current use of uni-directional fibre-composite patches as reinforcements [2]. Detailed results are presented in Section 3 for the case of elliptical orthotropic patches which are rigidly bonded to an infinite orthotropic plate.

To assess the second reinforcing effect (viz. the restraint on the crack opening), the assumption of a rigid bond must be discarded, as it would imply that the crack cannot open. It is now important to model more accurately the transfer of load through the adhesive layers in the immediate vicinity of the crack. But, if the load transfer length is sufficiently small compared with the lateral dimensions of the reinforcements, the restraint on the crack opening will not depend sensitively on the precise shape or extent of the reinforcements, so that both the plate and the reinforcements can now be assumed to be of infinite extent. With this idealization, one can use the results recently derived in [3] to estimate the crack extension force. This estimate is presented in Section 4, and it is compared in Section 5 with numerical results obtained by the finite element method [4]. Because our estimate is given as an explicit formula, it is particularly useful for assessing the influence of crack growth, or of variations in the shape of the reinforcements, which it would be laborious to study numerically. The present approach identifies a characteristic crack-length, which is related to, but not identical with, the lap-joint load-transfer length. The existence and the relevance of this characteristic length do not seem to have been recognized prior to the present writer's work [3].

A number of repairs of cracked metallic plates by bonded fibre-composite patches have been undertaken by Baker *et al.*, some of which are documented in [2]. I am indebted to Dr. Baker for suggesting a theoretical study of such repairs.

2. THE INCLUSION ANALOGY

2.1 *Notation and conventions*

We shall consider in Sections 2 and 3 an *uncracked plate* of uniform thickness $2t$, which is reinforced by two identical patches placed directly opposite one another across the plate, with one patch bonded to each face of the plate. This configuration is symmetrical about the plate's mid-plane. It will simplify the description of the problem to imagine the plate cut along its mid-plane and to speak as if we were considering a plate of thickness t with a single reinforcement of thickness t_R bonded on one side, but under an artificial restraint against out-of-plane bending. The results may be applied in practice to cases of one-sided reinforcement if it appears safe to neglect the actual out-of-plane bending.

The inclusion analogy will be formulated using the conventional two-dimensional idealization according to which the thickness-average of the in-plane stresses and displacements can be determined from the equations of generalized plane stress. The position of material points will be referred to an x, y coordinate system parallel to the plate's faces. Let the reinforcement cover an open region $\mathscr{D}$ bounded by a smooth curve $\mathscr{C}$ (Fig. 1). The plate can then be divided into an inner region $\mathscr{D}$ lying under the reinforcement, and an outer region complementary to $\mathscr{D}+\mathscr{C}$. For conceptual convenience, we shall suppose that $\mathscr{D}$ is a simply-connected region, whose boundary $\mathscr{C}$ lies strictly within the plate's outer boundary. The part of the plate complementary to $\mathscr{D}+\mathscr{C}$ will be denoted by $\mathscr{M}$, in view of its description later as the "matrix" surrounding an "inclusion" which occupies the region $\mathscr{D}$. $\mathscr{C}^+$, $\mathscr{C}^-$ will be used to indicate that the curve $\mathscr{C}$ is approached from inside (i.e. from $\mathscr{D}$), or from outside (i.e. from $\mathscr{M}$), respectively; $\mathbf{n}$ will denote the outward normal to $\mathscr{C}$ relative to $\mathscr{D}$, or equivalently, the inward normal to $\mathscr{C}$ relative to $\mathscr{M}$, as shown in Fig. 1.

Let $\sigma_{\alpha\beta}$, u_α denote respectively the stress components and the components of the displacement $\mathbf{u}$ in the plate. The usual conventions apply: (i) Greek subscripts stand for x or y, (ii) repeated Greek subscripts imply a summation, and (iii) a comma indicates partial differentiation with respect to the subscript which follows it. Parameters or field variables pertaining respectively to the reinforcement or the inclusion will be distinguished from the corresponding quantities pertaining to the base-plate by the addition of a subscript or superscript R or I. No

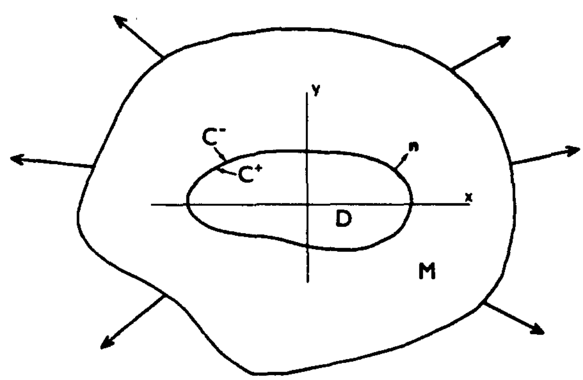

Fig. 1. An uncracked plate reinforced over an inner region $\mathscr{D}$ with external loads applied to its outer boundary, showing the notation used in Section 2. In Section 4 a crack will be introduced in the plate along a segment of the y-axis, under the reinforcement.

summation is implied over these affixes which will serve merely as labels. A symbol carrying no such label pertains to the plate.

The ultimate objective in Sections 2 and 3 is to determine the stresses $\sigma_{\alpha\beta}$ ($\mathscr{D}$) in the plate over the reinforced region $\mathscr{D}$, when prescribed external loads are applied to the plate's outer boundary, or equivalently, when a prescribed stress field would prevail in the plate without the reinforcement. No external forces will be applied to the reinforcement. The final objective will be reached by first calculating the stresses in an appropriately defined inclusion, and then determining how the load carried by the inclusion is shared between the plate and the reinforcement in the actual reinforced structure. It will be clear that the inclusion analogy could be formulated for arbitrary regions $\mathscr{D}$, and that the reinforcements could be allowed to overlap the plate and carry external loads, as in [1], but the above restrictions allow sufficient scope for the application which we have in mind.

2.2 *General formulation*

Consider the possibility of constructing an inclusion which will replace the reinforced portion $\mathscr{D}$ without altering the stress or displacement in $\mathscr{M}$, for any prescribed applied load. To simplify that construction, we shall assume that: (i) the adhesive layer transmits load from the plate to the reinforcement by exerting equal and opposite shear tractions to their adjoining faces; (ii) these shear tractions may be replaced by body forces having the same resultant, but distributed uniformly accross the thickness; and (iii) the average stress across the thickness can be determined by the theory of generalized plane stress. Then if $\mathbf{F}(x, y)$ denotes the shear traction per unit area at a point (x, y) on the reinforcement, the corresponding body forces in the plate and the reinforcement at that point are $-\mathbf{F}/t$ and $\mathbf{F}/t_R$, respectively, and the equations of equilibrium may be written as follows:

$$\sigma_{\alpha\beta,\beta}\,(\mathscr{D}) = F_\alpha/t, \tag{1}$$

$$\sigma^R_{\alpha\beta,\beta}\,(\mathscr{D}) = -F_\alpha/t_R. \tag{2}$$

Now suppose that a state of generalized plane stress prevails in a hypothetical plate of uniform thickness t_I having the shape $\mathscr{D}$ bounded by $\mathscr{C}$, with the stress components $\sigma^I_{\alpha\beta}$ *defined by*

$$t_I\sigma^I_{\alpha\beta}(\mathscr{D}) = t\sigma_{\alpha\beta}(\mathscr{D}) + t_R\sigma^R_{\alpha\beta}(\mathscr{D}). \tag{3}$$

From (1, 2) it follows that

$$\sigma^I_{\alpha\beta,\beta}(\mathscr{D}) = 0. \tag{4}$$

Thus, no body forces act within this hypothetical plate: the stress $\sigma^I_{\alpha\beta}$ must be due solely to tractions

$$T^I_\alpha(\mathscr{C}) = t_I\sigma^I_{\alpha\beta}(\mathscr{C}^+)\,n_\beta, \tag{5}$$

acting on its boundary $\mathscr{C}$.

To establish the inclusion analogy, it is sufficient to choose the constitutive properties of this hypothetical plate so as to fulfil the following boundary conditions on $\mathscr{C}$:

$$T^I_\alpha(\mathscr{C}) = -t\,\sigma_{\alpha\beta}\,(\mathscr{C}^-)\,n_\beta, \tag{6}$$

$$u^I(\mathscr{C}^+) = u(\mathscr{C}^-). \tag{7}$$

If this choice can be made, the reinforced portion $\mathscr{D}$ of the original plate can be replaced by the hypothetical plate, which can now be viewed as an "inclusion" having different constitutive properties from the surrounding "matrix".

So far, the constitutive properties of the plate and reinforcement are unrestricted, and no rule has been specified for evaluating the shear tractions transmitted by the adhesive. To proceed, we shall assume that (i) all components are homogeneous and linearly elastic, and (ii) the bond allows no relative displacement between the plate and the reinforcement. It is then natural to require that the same displacement field should prevail in the inclusion, i.e. we require

$$\mathbf{u}^I(\mathscr{D}) = \mathbf{u}\,(\mathscr{D}) = \mathbf{u}^R(\mathscr{D}), \tag{8}$$

so that boundary condition (7) is automatically satisfied. These assumptions are used from the start by Muki and Sternberg [1] who restrict their analysis to isotropic components. Having in mind applications involving uni-directional fibre-composite sheets, we shall assume that the plate and reinforcement are orthotropic, with their principal directions parallel to the x, y axes. The appropriate inclusion is then also orthotropic, with the same principal directions.

2.3 *Elastic constants of the equivalent inclusion* (*orthotropic plate and reinforcement*)

The elastic properties of an orthotropic plate under generalized plane stress can be specified by four constants, which are usually taken to be the principal Young's moduli E_x, E_y, the major Poisson's ratio $\nu(\equiv \nu_{xy})$, and the shear modulus $\mu(\equiv G_{xy})$. There are two Poisson's ratios, ν_{xy} and ν_{yx}, which are related by

$$\nu_{yx}E_x = \nu_{xy}E_y.$$

We have chosen ν_{xy} as the major Poisson's ratio, having in mind applications where the "fibre-direction" is parallel to the x-axis (then $E_x > E_y$). An isotropic plate can be considered as a special case of an orthotropic plate for which $E_x = E_y = E$, $\nu_{xy} = \nu_{yx} = \nu$, and μ is no longer an independent constant but is related to E, ν by $E = 2\mu(1+\nu)$.

The stress–strain equations for an orthotropic plate take the form

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \begin{bmatrix}A_x & \nu A_y & 0\\ \nu A_y & A_y & 0\\ 0 & 0 & 2\mu\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \epsilon_{xy}\end{bmatrix}, \tag{9}$$

with

$$A_x \equiv E_x/(1-\nu_{xy}\,\nu_{yx}),\; A_y \equiv E_y/(1-\nu_{xy}\nu_{yx}).$$

The elastic constants of the inclusion are obtained in their simplest form if A_x, A_y, ν, μ are used as the basic constants instead of E_x, E_y, ν, μ. (A similar, but less advantageous, simplification would occur for isotropic components if $A \equiv \mu/(1-\nu)$ and ν are used as the basic constants, instead of μ, ν as in [1]). From (3), which specifies the stress in the inclusion, we have

$$\sigma^I{}_{yy}t_I = \sigma_{yy}t + \sigma^R_{yy}t_R,$$

and from (8), which imposes equal displacements and thus equal strains, we have

$$\epsilon^I_{\alpha\beta} = \epsilon_{\alpha\beta} = \epsilon^R_{\alpha\beta}, \quad \text{in } \mathscr{D}.$$

Thus, we find

$$A^I_y t_I(\epsilon_{yy} + \nu_I\epsilon_{xx}) = A_y t(\epsilon_{yy} + \nu\epsilon_{xx}) + A^R_y t_R\,(\epsilon_{yy} + \nu_R\,\epsilon_{xx}).$$

This equation must hold for arbitrary ϵ_{xx}, ϵ_{yy}, so, by taking in turn $\epsilon_{xx} = 0$, then $\epsilon_{yy} = 0$, we derive the relations

$$A^I_y = (A_y t + A^R_y t_R)/t_I, \tag{10a}$$

$$\nu_I = (\nu A_y t + \nu_R A_y^R t_R)/(A_y t + A_y^R t_R), \tag{10b}$$

and similarly,

$$A_x^I = (A_x t + A_x^R t_R)/t_I, \tag{10c}$$

$$\mu_I = (\mu t + \mu_R t_R)/t_I. \tag{10d}$$

Equations (10) give the required elastic constants of the inclusion in terms of those of the plate and reinforcements. The inclusion-thickness t_I can be chosen arbitrarily: Muki and Sternberg set $t_I = t + t_R$; it will prove more convenient in the present context to choose

$$t_I = t,$$

so that (5, 6) imply the continuity of stresses across $\mathscr{C}$.

3. AN INFINITE PLATE WITH AN ELLIPTICAL REINFORCEMENT

3.1 *The elliptical inclusion*

The stress in an inclusion can be determined explicitly only for certain simple shapes and load distributions. Specifically, consider an infinite orthotropic plate with an elliptical inclusion whose boundary is described by

$$(x/a)^2 + (y/b)^2 = 1.$$

The plate is subjected to the following uniform biaxial tension at infinity

$$\sigma_{xx} = P, \sigma_{yy} = Q, \sigma_{xy} = 0, (x, y) \to \infty \tag{11}$$

with P, Q prescribed constants. It can be anticipated from the known results for the ellipsoidal inclusion [5], that the stress will be uniform inside an elliptical inclusion under generalized plane stress.

This expectation greatly simplifies the strategy for deriving an explicit solution. It leads us to suppose that the stress inside the inclusion is given by

$$\sigma_{xx}^I = P + p, \sigma_{yy}^I = Q + q, \sigma_{xy}^I = 0, \tag{12}$$

where p, q are constants which represent the "excess stress" in the inclusion relative to the applied stress at infinity. The choice $\sigma_{xy}^I = 0$ follows from the expectation of uniform stress and the fact that the symmetry with respect to the x- and y-axis implies

$$\sigma_{xy}^I = 0 \text{ on } x = 0 \text{ and } y = 0.$$

The excess stress will be determined in two steps. First, consider the region $\mathscr{M}$ as an infinite orthotropic plate with an elliptical hole which is loaded internally along $\mathscr{C}$ by the tractions

$$\mathbf{T} = -(pn_x, qn_y).$$

The resulting displacement $\mathbf{u}$ can be found by the method described in [6], Section 25, using the symmetry to set $u_x = 0$ on $x = 0$, $u_y = 0$ on $y = 0$. The important feature of the result is that

$$\mathbf{u}(\mathscr{C}^-) = (k_1 x, k_2 y),$$

where k_1, k_2 are linear combinations of p, q. This form of the displacement makes it simple to choose p, q so as to satisfy the continuity condition (6, 7), because the displacement in the inclusion due to the stress (12) is also of the form: $u_x^I(\mathscr{C}^+)$ proportional to x, $u_y^I(\mathscr{C}^+)$ proportional to y. One can thus derive two simultaneous linear equations for p, q, and this second step confirms the expected form (12) for the stress in the inclusion†.

The detailed expressions for p, q involve the roots of the characteristic equation

$$\lambda^4 + (E_x/\mu - 2\nu)\lambda^2 + E_x/E_y = 0. \tag{13}$$

These roots occur in complex conjugate pairs, and are never purely real (see [6], Section 19). They are purely imaginary if

$$(E_x/\mu - 2\nu)^2 > 4E_x/E_y.$$

As this condition generally holds for fibre composites, the equations determining p, q will be presented only for the case where (13) has the purely imaginary roots $\pm i\alpha$, $\pm i\beta$, $(\alpha > \beta)$. Then, with $t_I = t$, we find

$$\{(b/a)(\alpha+\beta)/E_x + 1/E_x^I\}p - \{(\alpha\beta - \nu)/E_x + \nu_I/E_x^I\}q = (1/E_x - 1/E_x^I)P - (\nu/E_x - \nu_I/E_x^I)Q, \tag{14}$$

$$-\{(\alpha\beta E_y)^{-1} - \nu/E_x + \nu_I/E_x^I\}p + \{(a/b)(\alpha+\beta)/(\alpha\beta E_y) + 1/E_y^I\}q = -(\nu/E_x - \nu_I/E_x^I)P + (1/E_y - 1/E_y^I)Q. \tag{15}$$

The method described in [6], Section 25, requires modification when the characteristic eqn (13) has coincident roots (see [6], Section 22). In particular, for the special case of an *isotropic plate*, (13) has double roots at $\pm$ i, but it can be verified (e.g. by using [7]) that (14, 15) give p, q correctly for that case if we set $\alpha = \beta = 1$; the reinforcement, and consequently the inclusion, can be either isotropic or orthotropic. It can also be verified, by inter-changing x and y, that (14, 15) hold for $b/a > 1$ as well as for $b/a \leq 1$.

Equations (14, 15) constitute two simultaneous linear equations for p and q whose solution is elementary. The stress in the inclusion then follows from (12).

3.2 *The stress at the prospective location of the crack*

It remains to determine how the load carried by the inclusion is shared between the plate and the reinforcement in the original structure. For the present application we only require $\sigma_{xx}(\mathscr{D})$. From the defining eqn (3), and the equality of strains imposed by (8), we find

$$\sigma_{xx}t = \sigma_{xx}^I t_I - (A_x^R \epsilon_{xx} + \nu_R A_y^R \epsilon_{yy}) t_R, \text{ in } \mathscr{D}.$$

Hence, with $t_I = t$ and using the inverse of (9), we derive

$$\sigma_{xx}(\mathscr{D}) = \sigma_{xx}^I \{1 - (t_R/t)(A_x^R - \nu_I \nu_R A_y^R)/E_x^I\} + \sigma_{yy}^I (t_R/t)(\nu_I A_x^R/E_x^I - \nu_R A_y^R/E_y^I). \tag{16}$$

Equation (16) simplifies considerably for the special case of an isotropic plate and reinforcement, both having the same Poisson's ratio, and this special case will be used below to illustrate the following two results which hold for the general case. First, there is no optimum stiffness for the reinforcement, in the sense that $|\sigma_{xx}(\mathscr{D})|$ decreases uniformly with increasing stiffness. Thus, from the point of view of reducing $\sigma_{xx}(\mathscr{D})$, the higher the stiffness of the reinforcement, the better. Secondly, in some cases there can be an optimum shape for which $|\sigma_{xx}(\mathscr{D})|$ is a minimum. The existence and the precise specification of that optimum shape depend on the ratio Q/P and on the reinforcement's stiffness relative to that of the plate.

†In fact, the stress in an elliptical inclusion is uniform under more general conditions than have been used here, as it could be expected from the corresponding results in [5].

3.3 *Influence of the reinforcement's shape and stiffness (isotropic case)*

Let E, E_R denote respectively the Young's modulus of the plate and the reinforcement, both of which are assumed to be isotropic and to have the same Poisson's ratio ν. Then, the equivalent inclusion is also isotropic with the same Poisson's ratio (i.e. $\nu_I = \nu = \nu_R$), and its Young's modulus is given by

$$E_I = E + E_R t_R/t. \tag{17}$$

Equation (16) reduces to

$$\sigma_{xx}(\mathcal{D}) = (E/E_I)\,\sigma^I_{xx}. \tag{18}$$

Let m, ρ denote respectively the *aspect ratio* (a/b) and *stiffness ratio* (E_I/E). Note that, for fixed b, the aspect ratio m varies from 1 to ∞ as the reinforcement changes from a disc of radius b to the strip $|y| < b$; for fixed a, m varies from 1 to 0 as the shape changes from a disc of radius a to the strip $|x| < a$; and, for fixed Et, the stiffness ratio ρ varies from 1 to ∞ as the stiffness $E_R t_R$ of the reinforcement changes from 0 to ∞. From (18), (12) and (14, 15) it follows that

$$\sigma_{xx}(\mathcal{D}) = (P + p)/\rho, \tag{19}$$

$$p = \{(\rho - 1)/D\}[2\rho m + 1)(P - \nu Q) + \{\nu + (1 - \nu)\rho\}(Q - \nu P)] \tag{20}$$

$$D = (2\rho/m + 1)(2\rho m + 1) - \{\nu + (1 - \nu)\rho\}^2.$$

To clarify the important features of this result, consider the following particular choices for the applied stress (11) at infinity.

(i) *Uniaxial tension parallel to the* x-*axis*: $P > 0, Q = 0$. For a fixed shape (m constant), the stress $\sigma_{xx}(\mathcal{D})$ under the reinforcement decreases monotonically with increasing reinforcement stiffness, the asymptotic behaviour for $\rho \gg 1$ being given by

$$\sigma_{xx}(\mathcal{D}) \simeq \frac{P}{\rho}(3 + \nu + 2m)/\{4 - (1 - \nu)^2\}.$$

Thus, there is no optimum stiffness for the reinforcement.

For fixed ρ, and $m \gg 1$,

$$\sigma_{xx}(\mathcal{D}) \simeq P,$$

i.e. there is no stress reduction when the reinforcement is elongated parallel to the tension, while for $m \ll 1$,

$$\sigma_{xx}(\mathcal{D}) \simeq \frac{P}{\rho}\{1 + m(1 - \nu)(1 + \nu - \nu\rho)(\rho - 1)/2\rho\}. \tag{21}$$

The remarkable feature of (21) is that for $(1 + \nu - \nu\rho) < 0$,
i.e. for

$$(Et/E_R t_R) < \nu,$$

the first-order term in m is negative, suggesting that there is an optimum value of m for which $\sigma_{xx}(\mathcal{D})$ takes a minimum value. By considering the derivative of $\sigma_{xx}(\mathcal{D})$ with respect to m, we find that this optimum shape is given by

$$m = \frac{[4 + (1 - \nu)(\nu\rho - 1 - \nu)\{4 - (1 - \nu)^2 - \nu(1 - \nu)/\rho\}]^{1/2} - 2}{\{4 - (1 - \nu)^2\}\rho - \nu(1 - \nu)}.$$

For example, with $\nu = 1/3$ there is an optimum shape whenever $\rho > 4$, i.e. whenever $E_R t_R > 3Et$. In particular, if $\rho = 10$, $\sigma_{xx}(\mathscr{D})$ decreases from $0.1\,P$ for $m = 0$, to minimum of $0.098\,P$ for $m = 0.027$ (when $b/a = 37$), and then it increases monotonically to $0.14\,P$ when $m = 1$, and towards the limiting value P as $m \to \infty$.

(ii) *Uniaxial tension parallel to the* y-*axis*: $\mathrm{P} = 0, \mathrm{Q} > 0$. Without the reinforcement, this tension would have no effect on a crack along $x = 0$, but with a reinforcement there is clearly a non-zero stress σ_{xx} ($\mathscr{D}$). The important feature is that the *sign* of this stress depends on the shape, while its *absolute value* decreases with increasing ρ, the asymptotic behaviour for $\rho \gg 1$ being given by

$$\sigma_{xx}(I) \simeq \frac{Q}{\rho}(1 - \nu - 2\nu m)/\{4 - (1 - \nu)^2\}.$$

Thus, for $\nu = 1/3$, σ_{xx} ($\mathscr{D}$) is positive when $m < 1$, and negative when $m > 1$. The choice of an optimum shape will therefore depend on the ratio Q/P.

(iii) *Simple shear at 45° to the* x, y-*axes*: $\mathrm{P} = -\mathrm{Q} > 0$. For all values of ρ there now exists an optimum aspect ratio m which is given by

$$m = [\{1 + (1 - \nu)(\rho - 1)\gamma/(2\rho)\}^{1/2} - 1]/\gamma,$$

$$2\rho\gamma \equiv \{4 - (1 - \nu)^2\}\rho^2 + (1 - \nu)\{(1 - 2\nu)\rho + 1\}.$$

For example, with $\nu = 1/3$, $\rho = 10$, we find that σ_{xx} ($\mathscr{D}$) decreases from $0.1\,P$ when $m = 0$ to a minimum of $0.087\,P$ when $m = 0.085$ ($b/a = 11.75$), then it increases to $0.14\,P$ when $m = 1$, and further towards $1.3\,P$ as $m \to \infty$. Note that the value of $\sigma_{xx}(\mathscr{D})$ for $m = 0$ and $m = 1$ is the same, to two significant figures, as for case (i).

(iv) *Hydrostatic tension*: $\mathrm{P} = \mathrm{Q} > 0$. There is no optimum shape or stiffness: for any value of ρ, σ_{xx} ($\mathscr{D}$) increases monotonically from P/ρ when $m = 0$ towards the value

$$(P/\rho)\,\{1 + (1 - \nu)(\rho - 1)\} \text{ as } m \to \infty.$$

4. THE CRACK EXTENSION FORCE

The next step is to introduce a crack by making a cut in the plate along $x = 0, |y| \leq c$, and allowing the normal tractions across this cut to relax to zero.

To obtain a realistic estimate for the *crack extension force G*, we must now allow for some relative motion between the plate and the reinforcement, otherwise the crack would not open so that G would be identically zero. We shall adopt a simple alternative assumption, *viz.* eqn (23) which leads to an explicit formula for the load transfer length, and an explicit estimate for G.

It should be emphasized that G cannot be determined analytically in closed form, not even for the special case of an isotropic plate and reinforcement, both having the same Poisson's ratio and both being of infinite extent. The best which can be done for that special case is to reduce the problem to the numerical solution of a Fredholm integral equation of the second kind, with a complicated kernel in the form of an integral over an infinite range involving a product of Bessel functions [8, 3]. The more general case has been studied numerically by a direct approach which leads to a two-dimensional integral equation whose kernel has a logarithmic singularity [9, 10], and by the finite element method [10, 11, 4]. The most extensive of these studies is that of Ratwani [10] who considers an infinite plate and reinforcement. Although his detailed results cannot be derived analytically, they can be well approximated by two upper bounds, one of which is an asymptote for $c \to 0$, the other an asymptote for $c \to \infty$, as shown in [3]. The estimate given below for G relies on these upper bounds. The reasoning used in [3] will be briefly recalled in a form adapted to our present concern with reinforcements which are smaller than the baseplate. We shall continue to consider the particular case of an elliptical reinforcement on an infinite plate, as in Section 3, for the sake of having a known constant value for σ_{xx} at the prospective location of the crack. That value will be denoted by σ_0, thus

$$\sigma_0 \equiv \sigma_{xx}(x = 0, |y| < c). \tag{22}$$

The results should be useful for other shapes of reinforcement which can be adequately approximated by an ellipse, at least for the purpose of determining σ_{xx} $(x=0, |y|<c)$ in the uncracked plate; these other shapes should also be symmetrical about $x=0$, so that σ_{xy} $(x=0)=0$, as for the case of an elliptical reinforcement.

4.1 *The load transfer length*

Recall from Section 2.2 that $\mathbf{F}(x, y)$ denotes the shear traction per unit area exerted by the adhesive layer on the reinforcement at the point (x, y). We continue to suppose that this surface traction can be replaced by a body force distributed uniformly across the thickness, but instead of the rigid-bond assumption (8), we shall now assume that

$$\mathbf{F}(x, y) = (\mu_A/t_A)\{\mathbf{u}(x, y) - \mathbf{u}^R(x, y)\}, \tag{23}$$

where the constants μ_A, t_A denote respectively the shear modulus and the thickness of the adhesive layer. This assumption has proved very successful as a basis for first approximations in the theory of lap-joints, see [12].

To estimate G we shall require the solution to the following problem. Suppose that an infinite reinforcing strip is bonded to two semi-infinite strips of the base-plate whose ends touch along $x=0$. These strips are taken to be of unit width so that they lie in the region $0 \leq y \leq 1$ of the plane. Next, suppose that the semi-infinite strips are prised apart by a pressure $\sigma_0 t$ applied to their ends at $x = \pm 0$, giving rise to a compressive stress

$$\sigma_{xx}(x = \pm 0, 0 \leq y \leq 1) = -\sigma_0.$$

The movement of these semi-infinite strips is resisted by adhesive shear tractions which transmit the load to the reinforcing strip. The problem is to calculate the work done in applying that pressure. The configuration is that of a lap-joint, but instead of the external force being applied to the base-plate at $x = \pm\infty$, it is being applied at $x = \pm 0$. The proposed calculation can be reduced to the solution of an ordinary differential equation if the variation of stress and displacement in the y-direction is ignored, as it is commonly done in dealing with lap-joints [12]. It is then readily shown that

$$F_x(x, y) = \Lambda\sigma_0 t\, e^{-\Lambda|x|},$$

$$\Lambda^2 = (\mu_A/t_A)\{(E_x t)^{-1} + (E_x^R t_R)^{-1}\}, \tag{24}$$

i.e. the adhesive shear tractions decay exponentially with distance from the joined ends. Λ^{-1} is known as the *lap-joint load-transfer length*. The work done, which will be denoted by G_∞, is given by

$$G_\infty = \sigma_0^2 t(t_A/\mu_A)\Lambda. \tag{25}$$

A more refined calculation (which allowed for variations in the y-direction and across the thickness) would lead to a more precise value of G_∞, but the above estimate is sufficiently accurate for our purposes.

4.2 *An upper bound for* G

When the crack is cut in, the load $2ct\sigma_0$ which was previously transmitted across the segment $x=0$, $|y|<c$ of the plate is partly transferred to the reinforcement and partly redistributed within the plate. The relative proportions in this repartition of load depend primarily on the crack length: the longer the crack, the greater the proportion of load which is transferred to the reinforcement. That part of the load which is redistributed within the plate gives rise to a stress singularity at the crack tips which can be characterized by a ***stress intensity factor K*** (see [13]). It is shown in [3] that the body force field corresponding to the shear tractions (23) does not affect the relation between G and K which is derived in [13]. Thus, for an isotropic plate under generalized plane stress,

$$G = K^2/E, \tag{26}$$

while for an orthotropic plate,

$$G = \{K^2/(2E_x E_y)^{1/2}\}\{(E_x/E_y)^{1/2} + E_y(E_x - 2\nu\mu)/2\mu E_x)\}^{1/2}. \qquad (27)$$

In what follows we shall therefore deal with either G or K, whichever is the simpler.

If we suppose that no load is transferred to the reinforcement, we shall certainly over-estimate K. This leads therefore to an upper bound for K which is given by

$$K_U = \sigma_0(\pi c)^{1/2}. \qquad (28)$$

A corresponding upper bound G_U for G is obtained from (26) or (27). These bounds will be closer to the actual values of G and K the shorter the crack, and in fact, for a reinforcement of infinite extent, they give an asymptote with first order contact in the limit $c \to 0$, [3]. For long cracks, however, these bounds greatly over-estimate the actual values, because most of the load $2ct\sigma_0$ is transferred to the reinforcement.

4.3 *An estimate for* G (*long cracks*)

Suppose first that both the reinforcement and the plate are of infinite extent. As the crack length $2c$ increases, an increasing amount of load has to be either transferred to the reinforcement or redistributed within the plate. The crack extension force G is necessarily an increasing function of crack length, but its value cannot exceed the force G_∞ on a semi-infinite crack. It is shown in [3] that G_∞ can be obtained from the calculation described above in Section 4.1, so that we may take G_∞ to be given by (25). The crack length $c = \Omega/\pi$ for which $G_U = G_\infty$ provides a characteristic length which is related to the load-transfer length Λ^{-1} given by (24): e.g. for an isotropic plate

$$\Omega = Et(t_A/\mu_A)\Lambda,$$

$$= \{Et(t_A/\mu_A)\ (1 + Et/E_R t_R)\}^{1/2}. \qquad (29)$$

The combination G_U for $\pi c \le \Omega$ and G_∞ for $\pi c \ge \Omega$ gives an upper bound, and therefore a *conservative estimate*, for G for any crack length. It is shown in [3], by direct comparison with the numerical results of [10], that this estimate is close to the actual value of G, the worst error (of up to 15%) occurring in the middle range $1/3 < \pi c/\Omega < 3$.

Returning to the present concern with reinforcements of finite size, we propose that the minimum of G_U and G_∞ will again provide a satisfactory estimate of G (though G_∞ is no longer necessarily an upper bound).

It would be difficult to supply a precise error analysis for this estimate; its accuracy is most readily assessed by comparison with numerical results for particular cases (Section 5). The estimate should be more accurate in giving the relative values for G for arbitrary variations of the parameters than it is in giving the particular value of G for a specific choice of parameters. This is especially valuable for assessing the influence of variations in crack length, or of variations in the aspect ratio of the reinforcement, which it would be laborious to study numerically.

5. COMPARISON WITH NUMERICAL RESULTS

To illustrate how the ideas presented above may be used in practice, we shall now consider a particular case for which some numerical results are available [4]†. This involves an isotropic plate which is sufficiently large that it may be considered to be of infinite extent. The plate is reinforced by orthotropic patches over a rectangular region $|x| \le 75$ mm, $|y| \le 25$ mm, and it is subjected to a uni-axial tension $\sigma_{xx} = P$ at infinity. A crack is introduced along $x = 0$, $|y| \le 19$ mm. The plate parameters are

$$E = 71\ \text{GPa},\ \nu = 1/3,\ t = 1.15\ \text{mm};$$

†I am indebted to Dr. R. Jones for access to the original numerical results which are presented in graphical form in [4].

the adhesive parameters are

$$\mu_A = 0.965\,\text{GPa},\ t_A = 0.1\,\text{mm};$$

the reinforcement parameters are

$$E_x^R = 208\,\text{GPa},\ E_y^R = 25\,\text{GPa},\quad \nu_R = 1/6\ t_R = n/8\,\text{mm},$$

where n is the number of plies in the reinforcement (the reinforcement's shear modulus μ_R will not be required).

The first step is to estimate the stress σ_0 at the prospective location of the crack. It can be expected that the assumption of a rigid bond is appropriate over an "inner region", the width of the margin between that inner region and the actual boundary of the reinforcement being comparable with the load-transfer length Λ^{-1} for lap-joints. This expectation is confirmed by the numerical results shown in Fig. 15 of [11]. In the present case $\Lambda^{-1} \simeq 2$ mm (and this is a typical value in applications [2]) so that it may be neglected. It also seems reasonable to suppose that for the purposes of estimating σ_0 the rectangular reinforcement may be replaced by an elliptical one having the same aspect ratio $m \equiv a/b = 3$. Thus, we shall use the results given in Section 3. The appropriate inclusion parameters depend on the thickness of the reinforcement, i.e. on n, the number of plies. The largest value of n considered in [4] is 6, for which the appropriate inclusion parameters are found from (10) to be

$$E_x^I = 207.3\,\text{GPa},\ E_y^I = 92.3\,\text{GPa},\ \nu_I = 0.3,\ (t_I = t).$$

Solving (14, 15) for p, q and using (12, 16) we derive $\sigma_0 = 0.67\,P$.

The next step is to calculate the characteristic length Ω. We find $\Omega = 3.6$ mm, so that $\pi c/\Omega \gg 1$ for the present value of $c = 19$ mm. Thus the appropriate estimate for K is $K_\infty = \sigma_0\Omega^{1/2}$. Our estimated reduction in K, relative to the value $K = P(\pi c)^{1/2}$ which would prevail without the reinforcements, is found to be 83.5%, while the value given in [4] is 80.5%.

Other quantities reported in [4] can also be estimated by the present approach. For example, the maximum stress σ_{xx}^R (max) in the reinforcement occurs at $x = 0$, $y = 0$, and if $\pi c \gg \Omega$, as in the present case, this stress can be estimated by supposing that all the load carried locally by the inclusion is now carried by the reinforcement, so that

$$\sigma_{xx}^R\,(\text{max}) \simeq \sigma_{xx}^I\ t/t_R = 2.6\,P.$$

This is exactly the value given in [4].

The above calculations, repeated for $n = 1$, give a reduction in K of 72% and σ_{xx}^R (max) = 10.7 P, while the corresponding results in [4] are 66.5% and 9.9 P, respectively. The agreement with the numerical results of [4] is considered to be satisfactory, in view of the replacement of a rectangular patch by an elliptical one.

6. CONCLUSION

Bonded reinforcements reduce the crack extension force in two distinct ways: first, they reduce the stress in an uncracked plate at the prospective location of the crack; and secondly, they restrain the opening of the crack when it is introduced in that reduced stress field. These two effects can be adequately estimated by using two different modelling assumptions for the transfer of load through the adhesive. One practical advantage of making this distinction is that it can be useful to depend only on the first effect, the stress reduction, which relies on the successful transfer of load at the edge of the reinforcement. This stress reduction would not be affected if there were no adhesive in a central region under the reinforcement, or if the adhesive in that inner region fails when the crack is introduced, provided that the width of the region of adhesion around the edge of the reinforcement is larger than the load-transfer length. The reduction in crack extension force for short cracks (i.e. with $\pi c/\Omega \ll 1$) relies almost entirely on this first effect.

The stress reduction in an uncracked plate can be estimated by assuming a rigid bond and using an inclusion analogy. The elliptical inclusion has been studied analytically and two important points emerge from the results:

(i) In some cases there can be an optimum shape, or aspect ratio, for the reinforcement. However, this optimum does not seem to be very sharp and it appears sufficient to guard against non-optimal shapes. Thus, with an applied uniaxial tension $\sigma_{xx} = P$, a reinforcement which is elongated parallel to the x-axis (in the limit, a stringer) does not reduce the stress at the prospective location of the crack along $x = 0$, so that its restraining action is due entirely to the second effect noted above. (Nevertheless, quite substantial reductions in crack extension force can be achieved by bonded stringers placed close to the crack tip, if the adhesive remains elastic [14]).

(ii) A tensile or compressive stress parallel to the y-axis can lead to a tensile stress σ_{xx} at the prospective crack location, the main determining factor being again the shape of the reinforcement. Without the reinforcements, a normal stress parallel to the y-axis would have no effect on a crack lying along the y-axis, but it is important not to ignore such stresses when a plate is reinforced.

The second effect, namely the restraining action of the reinforcement when a crack is introduced, has been sufficiently discussed in [3], where it was noted that the mechanical properties of bonded reinforcements in this respect (e.g. their fatigue strength) can be adequately assessed from those of lap-joints. The present work, in combination with [3], provides a useful conceptual framework for detailed experimental or numerical studies, and even for the practical design, of bonded reinforcements for cracked plates.

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